4.5 Modelling With Algebra

Algebra is an efficient way to express mathematical ideas. Algebraic modelling is one of many ways to solve a problem. The best method often depends on the type of problem and the preference of the problem solver.

Investigate

How can you use algebraic modelling to solve problems?

1. Work with a partner or in a small group. One person will be the magician, and the others will be the audience. Have all students close their textbooks except the magician.

Magician's Instructions to the Audience

2. Tell the audience to pick a number from 1 to 10 and write it down so everyone but the magician can see it. Instruct the audience to carry out the following arithmetic steps, out of view of the magician.

- **a)** Take the number and double it.
- **b)** Add 50.
- **c)** Triple the result.
- **d)** Subtract 100.
- **e)** Divide this value in half.
- **f)** Write down the final result and show it to the magician. Show *only* the final result.

Magician's Secret Steps to Find the Number

3. Look at the result. Announce the following: "I will now use the magic of algebra to determine your number!" Add a magic word or phrase if you like. Try not to let the audience know that you are doing the following calculations mentally.

- **a)** Subtract 25 from the number you see.
- **b)** Divide by 3. The result is the number the audience picked.
- **c)** Announce the number to the amazement of your audience. Take a bow!

4. Repeat the magic trick with a different number.

Time to Reveal the Secret

5. Everyone in the group should now open their books and look at all of the instructions. As a group, discuss how you think this magic trick works.

6. **Reflect**

a) Is this trick really magic? Explain.

b) Will this trick work for numbers greater than 10? What about negative numbers? Explain.

c) How can you use algebra to create a magic trick of your own?

Example 1 Apply Algebraic Modelling to a Payroll Problem

Mr. Skyvington operates a variety store with his two sons, Jerry and Koko.

- Jerry makes twice as much as Koko, who only works part-time.
- Mr. Skyvington makes \$200 per week more than Jerry.
- The total weekly payroll is \$1450.

How much does each family member earn per week?

Solution

Use algebra to model and solve the problem.

Write an expression that describes each person's earnings. Let k represent Koko's earnings.

Worker	Expression	Explanation
Koko	k	Koko's earnings are unknown.
Jerry	$2k$	Jerry makes twice as much as Koko.
Mr. Skyvington	$2k + 200$	Mr. Skyvington earns \$200 more than Jerry.
Total	\$1450	

Use a table to organize your thinking.

Write an equation that relates these expressions to the total payroll.

$$\underbrace{k}_{\text{Koko's earnings}} + \underbrace{2k}_{\text{Jerry's earnings}} + \underbrace{2k + 200}_{\text{Mr. Skyvington's earnings}} = \underbrace{1450}_{\text{Total payroll}}$$

Solve the equation for k.

$$\begin{aligned} k + 2k + 2k + 200 &= 1450 \\ 5k + 200 &= 1450 \\ 5k + 200 - 200 &= 1450 - 200 && \text{Subtract 200 from both sides.} \\ 5k &= 1250 \\ \frac{5k}{5} &= \frac{1250}{5} && \text{Divide both sides by 5.} \\ k &= 250 \end{aligned}$$

I can check to see if these answers add to give the correct total:

Koko:	\$250
Jerry:	\$500
Mr. Skyvington:	\$700
Total	\$1450

The three wages add to the correct total.

The solution $k = 250$ means that Koko earns \$250 per week. Substitute into the other expressions to find how much Jerry and Mr. Skyvington earn.

Jerry:

$$\begin{aligned} & 2k \\ &= 2(250) \\ &= 500 \end{aligned}$$

Jerry earns \$500 per week.

Mr. Skyvington:

$$\begin{aligned} & 2k + 200 \\ &= 2(250) + 200 \\ &= 500 + 200 \\ &= 700 \end{aligned}$$

Mr. Skyvington earns \$700 per week.

Example 2 Apply Algebraic Modelling to an Earnings Problem

Uma works at a ballpark, selling peanuts. She is paid \$6/h plus a 50¢ commission for every bag of peanuts she sells.

a) Find Uma's earnings if she sells 42 bags of peanuts during a 4-h shift.

b) How many bags of peanuts must she sell to earn \$100 in 7 h?

Solution

a) Uma is paid in two ways:
- for the length of time she works (hourly wage)
- for the number of bags of peanuts she sells (commission)

Write an expression for each. Then, write a formula that models Uma's total earnings.

Earnings	Variable	Expression	Explanation
Hourly Wage	h	$6h$	Uma makes \$6/h.
Commission	p	$0.5p$	Uma earns 50¢ per bag of peanuts.
Total Earnings	E	$6h + 0.5p$	Add wage and commission to get total earnings.

The following formula describes Uma's earnings:
$E = 6h + 0.5p$

Substitute $h = 4$ and $p = 42$ to find Uma's total earnings.

$$\begin{aligned} E &= 6(4) + 0.5(42) \\ &= 24 + 21 \\ &= 45 \end{aligned}$$

Uma earns \$45 if she sells 42 bags of peanuts in 4 h.

b) To find the number of bags of peanuts Uma must sell to make \$100 in 7 h, rearrange the formula to express p in terms of E and h.

$$\begin{aligned} E &= 6h + 0.5p \\ E - 6h &= 6h + 0.5p - 6h \quad \text{Subtract } 6h \text{ from both sides.} \\ E - 6h &= 0.5p \end{aligned}$$

Literacy Connections

A wage is a payment that depends on the length of time worked. A commission is a payment based on the number of items sold or a percent of total sales.

Method 1: Divide by 0.5

$$E - 6h = 0.5p$$
$$\frac{E - 6h}{0.5} = \frac{0.5p}{0.5}$$
$$\frac{E - 6h}{0.5} = p$$
$$\text{or } p = \frac{E - 6h}{0.5}$$

Substitute $E = 100$ and $h = 7$.

$$p = \frac{100 - 6(7)}{0.5}$$
$$= \frac{100 - 42}{0.5}$$
$$= \frac{58}{0.5}$$
$$= 116$$

Method 2: Multiply by 2

$$E - 6h = 0.5p$$
$$2(E - 6h) = 2(0.5)p$$
$$2(E - 6h) = p$$
$$\text{or } p = 2(E - 6h)$$

I can also solve for p by multiplying by 2, because 0.5 is $\frac{1}{2}$.
$2 \times 0.5p = p$

Substitute $E = 100$ and $h = 7$.

$$p = 2[(100) - 6(7)]$$
$$= 2(100 - 42)$$
$$= 2(58)$$
$$= 116$$

Uma must sell 116 bags of peanuts to earn \$100 in a 7-h shift.

Compare these two equations: $p = \frac{E - 6h}{0.5}$ and $p = 2(E - 6h)$.

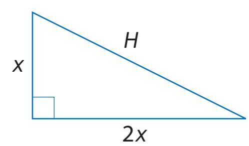
These are equivalent equations. Why do you think the second equation may be a little easier to use?

Example 3 Compare Algebraic Modelling With Other Strategies

Tan is designing a Japanese rock garden in the shape of a right triangle so that the second-shortest side is twice the length of the shortest side. The area of the garden must be 30 m^2. What are the three side lengths of Tan's garden, to the nearest tenth?

Solution

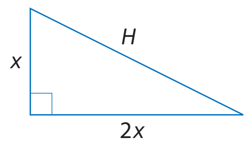
Method 1: Algebraic Model



The garden is in the shape of a right triangle. The second-shortest side is twice the length of the shortest side.

Let x represent the shortest side and $2x$ the second-shortest side.

Let H represent the hypotenuse.



Apply the formula for the area of a triangle.

$$A = \frac{1}{2}bh$$

$$30 = \frac{1}{2}(2x)(x) \quad \text{Substitute } b = 2x \text{ and } h = x.$$

$$30 = \frac{1}{2}(2x^2)$$

$$30 = x^2$$

$$\sqrt{30} = \sqrt{x^2} \quad \text{Take the square root of both sides.}$$

$$5.48 \doteq x$$

The length of the shortest side is 5.5 m, to the nearest tenth. Double this to find the length of the second-shortest side.

$$\begin{aligned} & 2x \\ = & \ 2(5.5) \\ = & \ 11 \end{aligned}$$

The two shorter sides are 5.5 m and 11 m. Apply the Pythagorean theorem to find the hypotenuse.

$$H^2 = 5.5^2 + 11^2$$

$$H^2 = 30.25 + 121$$

$$H^2 = 151.25$$

$$\sqrt{H^2} = \sqrt{151.25} \quad \text{Take the square root of both sides.}$$

$$H \doteq 12.3$$

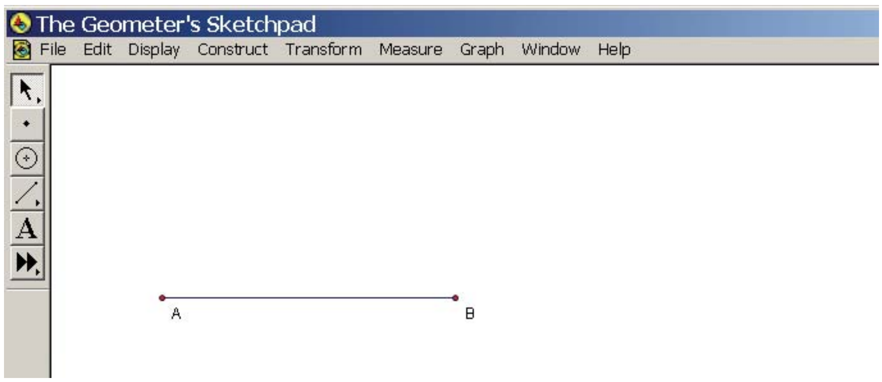
The algebraic model shows how measurement, algebra, and the Pythagorean Theorem are all connected. That's cool!

The three side lengths of Tan's garden are 5.5 m, 11 m, and 12.3 m.

Method 2: Construct a Graphical Model With *The Geometer's Sketchpad*®

Open a new sketch and construct a right triangle that meets the requirements.

- Construct a horizontal line segment AB.



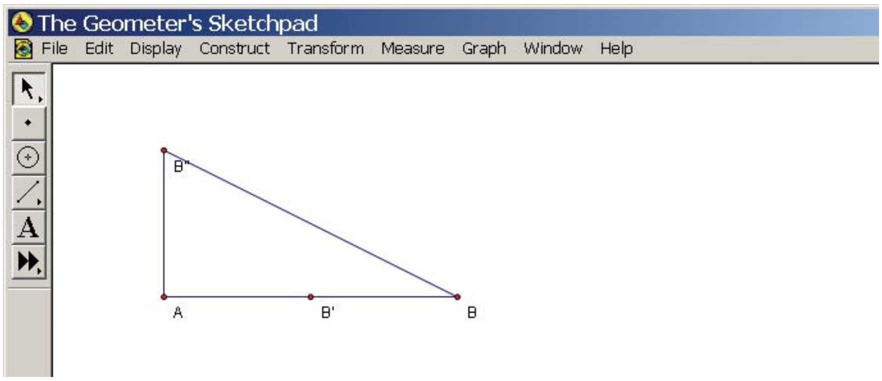
Technology Tip

Hold the Shift key down while you drag to make vertical and horizontal line segments.

Remember to deselect before making new selections. You can deselect by clicking anywhere in the white space with the **Selection Arrow Tool**.

- Select point A. From the **Transform** menu, choose **Mark Center**.
- Select line segment AB and point B. From the **Transform** menu, choose **Dilate**. Dilate the segment in the ratio 1:2. Click on **Dilate**.

- Select AB′. From the **Transform** menu, choose **Rotate**. Rotate the segment and point by 90°. Click on **Rotate**.
- Construct a segment that connects B and B″.

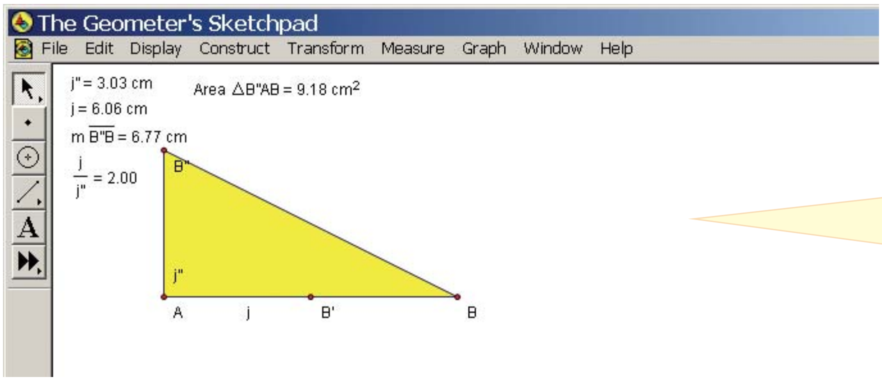


Measure the lengths of the three sides and verify that the two shorter sides are in the ratio 1:2.

- Select the three sides and, from the **Measure** menu, choose **Length**.
- From the **Measure** menu, choose **Calculate**. Divide the longer measure by the shorter measure.

Next, measure the area of the triangle.

- Select the three vertices, and from the **Construct** menu, choose **Triangle Interior**.
- From the **Measure** menu, choose **Area**.



Once I set up a geometric model with *The Geometer's Sketchpad*®, I can explore all kinds of relationships. I wonder how perimeter and area are related in this type of triangle.

- Adjust the size of the triangle so that its area is as close to 30 cm^2 as possible. Select and drag any of the vertices. Verify that the triangle has the following properties:
  – it is a right triangle
  – the second-shortest side is twice the length of the shortest side

The lengths of the three sides give the dimensions of Tan's garden. The measures are in centimetres in the sketch, but Tan's garden is measured in metres. The lengths of the three sides of Tan's garden are 5.5 m, 11 m, and 12 m.

Key Concepts

- Algebraic modelling is one method that can be used to describe mathematical situations and solve problems.
- Many problems can be solved using more than one method.

Communicate Your Understanding

C1 Rufio is 5 years older than his sister, Hanna. The sum of their ages is 37. Which equation can you use to find their ages? Explain why.

A $5h = 37$ **B** $h + 5h = 37$

C $h + 5 = 37$ **D** $h + (h + 5) = 37$

C2 One summer, Brittany had a paper route and a babysitting job. She made twice as much money babysitting as she did delivering papers. Altogether she made $800 that summer. Which equation can you use to find how much Brittany earned delivering papers? Explain.

A $2p = 800$ **B** $p + 2p = 800$

C $p + 2 = 800$ **D** $p + (p + 2) = 800$

C3 Asraf sells computers. He is paid $12/h, plus a 10% commission on sales. Which expression describes Asraf's total earnings? Explain.

A $12h + 0.1$ **B** $12 + 0.1s$

C $12h + 0.1s$ **D** $12h + 10s$

Practise

For help with questions 1 to 5, see Examples 1 and 2.

1. Write an algebraic expression to represent each description.

a) triple a number

b) four more than a number

c) half a number

d) five less than double a number

2. Write an equation to represent each sentence. Explain your choice of variable and what it represents in each case.

a) four times a number is 112

b) a perimeter increased by 12 is 56

c) five more than triple a number is 29

d) the sum of two consecutive integers is 63

Literacy Connections

Consecutive means one after the other. For example:

- 3, 4, and 5 are consecutive integers
- q, r, and s are consecutive letters

3. Solve each equation in question 2, and explain what the answer means.

4. Estaban is 6 years older than his brother Raoul. The sum of their ages is 38. How old are the brothers?

5. Two friends enter a trivia challenge as a team. Fayth scored 200 more points than Jamal. As a team, they collected a total of 2250 points. How many points did each friend earn?

Connect and Apply

6. Natalie, Chantal, and Samara play together as a forward line on a hockey team. At the end of the season, Chantal had scored eight more goals than Natalie, while Samara had scored twice as many goals as Natalie. The three girls scored a total of 52 goals. How many goals did each girl score?

7. Kyle sells used cars. He is paid \$14/hour plus an 8% commission on sales. What dollar amount of car sales must Kyle make to earn \$1200 in a 38-h work week?

8. **Chapter Problem** At the season finale, you present the winner of Canadian Superstar with a recording-and-tour contract. The contract states that the winner will be paid \$5000 per month while on tour plus \$2 per CD sold.

   a) Write an equation that relates total earnings in terms of the number of months, m, on tour and the number, n, of CDs sold.

   b) How much will the winner earn after the first month if 500 CDs are sold?

   c) Suppose after the third month on tour the new recording artist has earned a total of \$74 000. How many CDs were sold?

   d) In Canada, a record album or CD achieves gold status once it sells 50 000 units. How much will the artist make if the CD goes gold after 6 months of touring?

Did You Know?

In March 2003, *Dark Side of the Moon*, by Pink Floyd, achieved double diamond status in Canada for selling over 2 000 000 units.

9. The sum of three consecutive integers is 54. Find the numbers.

10. The sum of two consecutive even integers is -134. Find the numbers.

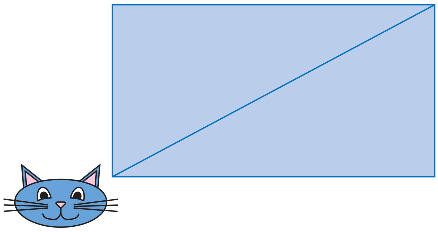
11. A circular garden has a diameter of 12 m. By how much should the diameter be increased to triple the area of the garden?

12. Refer to question 11.

    a) Solve the problem using a different method.

    b) Compare the two methods. Identify at least one advantage and one disadvantage of each approach.

13. The length of Laurie's rectangular swimming pool is triple its width. The pool covers an area of 192 m^2.



a) If Laurie swims across the diagonal and back, how far does she travel?

b) At the same time Laurie starts swimming, her cat walks one lap around the edge of the pool. Laurie can swim $\frac{3}{4}$ as fast as her cat can walk. Who will return to the starting point first? Justify your answer.

14. Refer to the magic trick in the Investigate. Create a magic trick of your own. Try your trick out on a friend or family member.

Achievement Check

15. Paloma works part-time, 4 h per day, selling fitness club memberships. She is paid \$9/h, plus a \$12 commission for each 1-year membership she sells.

a) Write an algebraic expression that describes Paloma's total earnings.

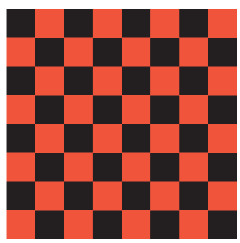
b) Find the amount Paloma makes in 8 h when she sells seven memberships.

c) How many memberships does Paloma need to sell to earn \$600 in a 24-h workweek?

d) Paloma notices that her sales have a pattern: for the first 12 h of the week she sells an average of two memberships per hour and for the last 12 h of the week she sells an average of three memberships per hour. Use an organized method (e.g., chart, graph, equations) to determine when Paloma will reach a special \$900 earnings goal.

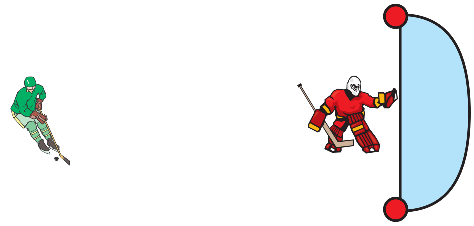
Extend

16. A checkerboard has 64 congruent squares. Suppose a checkerboard has a diagonal length of 40 cm. Find the area of each square on the board.



17. Johnny is directly in front of Dougie, who is playing goalie, as shown.

Johnny is 2.8 m from both goal posts. He is also three times as far from Dougie as Dougie is from either post.



a) How wide is the net?

b) Describe how you solved this problem.

c) Discuss any assumptions you had to make.

18. Johannes Kepler (1571–1630) was a German astronomer who noticed a pattern in the orbits of planets. The table shows data for the planets known when Kepler was alive.

Planet	Radius of Orbit (AU)*	Period of Orbit (Earth Days)
Mercury	0.389	87.77
Venus	0.724	224.70
Earth	1.0	365.25
Mars	1.524	686.98
Jupiter	5.200	4332.62
Saturn	9.150	10759.20

**AU, or astronomical unit, is the mean distance from Earth to the Sun, 1.49×10^8 km.*

a) Kepler conjectured that the square of the period divided by the cube of the radius is a constant. Copy the table. Add another column and compute the value of the square of the period divided by the cube of the radius for each planet. Then, find the mean of these values to find Kepler's constant.

b) Write a formula for the relationship that Kepler found. This is called Kepler's Third Law.

c) In 1781, William Herschel discovered the planet Uranus, which has a period of 30 588.70 days. Use Kepler's Third Law to determine the radius of Uranus's orbit.

d) In 1846, the planet Neptune was discovered. Neptune's orbital radius is 30 AU. Use Kepler's Third Law to find the orbital period of Neptune.

e) The planet Pluto has an orbital radius of 39.5 AU and a period of 90 588 days. Does Pluto satisfy Kepler's Third Law? Explain.

f) Investigate Kepler's other two laws of planetary motion. Write a brief report of your findings.

19. **Math Contest** The mass of a banana plus its peel is 360 g. The mass of the banana is four times the mass of the peel. What is the mass of the peel?

20. **Math Contest** Given that $y = 4x + 1$ and $z = 5x - 3$, and the value of z is 7, what is the value of y?

A -2 **B** -9 **C** 2 **D** 9 **E** 29